Motion of an Artificial Earth Satellite under the Influence of the Sun and Moon

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During the flight of artificial earth satellites outside the terrestrial atmosphere the principal disturbances are caused by the nonsphericity of the earth and the attraction of the sun and moon.

The effect of the nonsphericity of the earth on the motion of an artificial earth satellite has been studied sufficiently well. In the present work the disturbing effect of the sun and moon is examined. These bodies are considered to be physical points moving in elliptical orbits.

Lagrange's equations for the osculating elliptic elements of a satellite orbit have the following form (1):[1]

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial W}{\partial M}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2e}\frac{\partial W}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial W}{\partial \omega}$$

$$\frac{di}{dt} = \frac{\cot i}{na^2\sqrt{1-e^2}}\frac{\partial W}{\partial \omega} - \frac{1}{\sin i}\frac{1}{na^2\sqrt{1-e^2}}\frac{\partial W}{\partial \Omega}$$

$$\frac{dM}{dt} = n - \frac{2}{na}\frac{\partial W}{\partial a} - \frac{1-e^2}{na^2e}\frac{\partial W}{\partial e} \qquad [1]$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial W}{\partial e} - \frac{\cot i}{na^2\sqrt{1-e^2}}\frac{\partial W}{\partial i}$$

$$\frac{d\Omega}{dt} = \frac{1}{\sin i}\frac{1}{na^2\sqrt{1-e^2}}\frac{\partial W}{\partial i}$$

$$n = \frac{\sqrt{fm_0}}{a\sqrt{a}}$$

Here f is the constant of gravitation, a the semimajor axis, e the eccentricity, i the inclination of the orbit, ω the longitude of the perigee, Ω the longitude of the ascending node of the orbit, M the mean anomaly of the satellite orbit, m_0 the mass of the earth, and W the disturbing function which is defined by the formula

$$W = fm_L\left(\frac{1}{\Delta_L} - \frac{r\cos\psi_L}{r_L^2}\right) + fm_S\left(\frac{1}{\Delta_S} - \frac{r\cos\psi_S}{r_S^2}\right)$$

where m_L and m_S are the masses corresponding to the moon and the sun, $x_L, y_L, z_L, x_S, y_S, z_S$ are the coordinates of the moon and the sun in a coordinate system with its origin at the center of mass of the earth.

$$r_i^2 = x_i^2 + y_i^2 + z_i^2$$

$$\Delta_i^2 = (x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2$$

$$rr_i\cos\psi_i = xx_i + yy_i + zz_i$$

(here and in the future $i = L,S$), x, y, and z are the satellite coordinates expressed in terms of the elliptic elements (2):

$$x = a[P_x\cdot(\cos E - e) + \sqrt{1-e^2}\,Q_x\sin E]$$

$$y = a[P_y\cdot(\cos E - e) + \sqrt{1-e^2}\,Q_y\sin E] \qquad [2]$$

$$z = a[P_z\cdot(\cos E - e) + \sqrt{1-e^2}\,Q_z\sin E]$$

$$r = a(1 - e\cos E)$$

P_x, P_y, P_z, Q_x, Q_y, and Q_z are coefficients depending upon ω, Ω, i, and E is the eccentric anomaly.

Expanding the disturbing function according to Legendre's polynomials (3) we obtain

$$\frac{1}{\Delta_i} - \frac{r\cos\psi_i}{r_i^2} = \frac{1}{r_i}\left(\frac{r}{r_i}\right)^2\sum_{k=2}^{\infty}\left(\frac{r}{r_i}\right)^{k-2}P_k(\cos\psi_i)$$

$$W_i = n^2\left(\frac{a}{c}\right)^3\delta_i^2\left(\frac{a_i}{r_i}\right)^3 r^2\sum_{k=2}^{\infty}\left(\frac{r}{r_i}\right)^{k-2}P_k(\cos\psi_i)$$

where c is the polar radius of the earth and δ_i are infinitesimally small parameters

$$\delta_i^2 = \frac{m_i}{m_0}\left(\frac{c}{a_i}\right)^3$$

$$\delta_L^2 = 5.6\cdot10^{-8} \qquad \delta_S^2 = 2.6\cdot10^{-8}$$

We shall look for the solution of Eqs. [1] in the form

$$\mathfrak{E} = \mathfrak{E}_0 + \delta_L^2\mathfrak{E}_2^L + \delta_S^2\mathfrak{E}_2^S$$

where $\mathfrak{E}$ is any one of the osculating elements. Let us introduce the infinitesimals δ_L^2 and δ_S^2, the product of which is small enough to be neglected. That is, we shall study separately the disturbing effect of the moon and sun on the satellite.

The motion of a satellite is studied in the range

$$1.2c \leqslant r \leqslant 10c \qquad [3]$$

where the limits were taken such that the height of the perigee is about 10^3 km over the surface of the earth, and the apogee is approximately 50 to $100\cdot10^3$ km. The numerical estimates are carried out for a height of $70\cdot10^3$ km.

In this range $r/r_L \leqslant \frac{1}{6}$, $r/r_S \leqslant 4.3\cdot10^{-4}$. Since the ratio r/r_i is small, in the sum Σ_S only one component is retained with P_2; and in the sum Σ_L the components with P_2, P_3, and P_4 are retained. Only the constant part of the term in P_4 is taken. This part gives the secular perturbations in the elements. The remainder terms have the form

$$R_S = \sum_{k=3}^{\infty}\left(\frac{r}{r_S}\right)^{k-2} P_k(\cos\psi_S) \leqslant \frac{\left(\dfrac{r}{r_S}\right)}{1-\dfrac{r}{r_S}} < 4.3\cdot10^{-4}$$

$$R_L = \sum_{k=5}^{\infty}\left(\frac{r}{r_L}\right)^{k-2} P_k(\cos\psi_L) \leqslant \frac{\left(\dfrac{r}{r_L}\right)^3}{1-\dfrac{r}{r_L}} < 5.5\cdot10^{-3}$$

In Eqs. [1], when we discard the series R_i, we do not take into account the components causing disturbances in the elements on the order of 10^{-7}.

In investigating the perturbations due to the moon, the reference plane is taken as the plane of the orbit of the moon so that $z_L = 0$, $\omega_L = 0$ (the axis x is directed to the perigee). Then $\cos\psi_L = \cos u\cos(\Omega - v_L) - \sin u\cos i\sin(\Omega - v_L)$, where v_L is the true anomaly of the moon and $u = v + \omega$ is the argument of the latitude of the satellite. Since r/r_L is assumed small

$$W_L = n^2(a/c)^3\delta_L^2(a_L/r_L)^3r^2\,[(\tfrac{3}{2}\cos^2\psi_L - \tfrac{1}{2}) + (r/r_L)(\tfrac{5}{2}\cos^3\psi_L - \tfrac{3}{2}\cos\psi_L) + (r/r_L)^2(\tfrac{35}{8}\cos^4\psi_L - \tfrac{15}{4}\cos^2\psi_L + \tfrac{3}{8})] \qquad [4]$$

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[1] Numbers in parentheses indicate References at end of paper.

In investigating the perturbations due to the sun, it is more convenient to take the reference plane as the plane of the ecliptic and to direct the x axis to the perigee of the orbit of the sun, so that $z_S = 0$ and $\omega_S = 0$. Then $\cos\psi_S = \cos u \cos(\Omega - v_S) - \sin u \cos i \sin(\Omega - v_S)$ where v_S is true anomaly of the sun. The disturbing function expressing the action of the sun on the satellite will have the following form:

$$W_S = n^2(a/c)^3\delta_S^2(a_S/r_S)^3r^2(\tfrac{3}{2}\cos^2\psi_S - \tfrac{1}{2}) \qquad [5]$$

For the integration of the equations of [1] it is convenient to represent W_i in the form of the function of the satellite eccentric anomaly E. Using Eqs. [2], we express $\cos\psi_i$ in terms of E:

$$\cos\psi_i = (a/r)[(\cos E - e)(P_x \cos v_i + P_y \sin v_i) + \sqrt{1-e^2}\sin E\cdot(Q_x \cos v_i + Q_y \sin v_i)]$$

Substituting $\cos\psi_i$ into Eqs. [4] and [5] we obtain

$$W_i = n^2(a/c)^3\delta_i^2 \frac{a^2}{2}\sum_{j=0}^{3}(A_j^{(i)}\cos jE + B_j^{(i)}\sin jE)$$

where

$$A_j^{(S)} = (a_S/r_S)^3[a_{j0} + a_{j2}\cos 2v_S + \alpha_{j2}\sin 2v_S] \qquad (j = 0, 1, 2)$$

$$B_j^{(S)} = (a_S/r_S)^3[b_{j0} + b_{j2}\cos 2v_S + \beta_{j2}\sin 2v_S] \qquad (j = 1, 2)$$

$$A_3^{(S)} = B_3^{(S)} = 0$$

$$A_0^{(L)} = (a_L/r_L)^3[(a_{00} + (a^2/r_L^2)a_{00}^{(3)}) + (a_{02} + (a^2/r_L^2)a_{02}^{(3)})\cos 2v_L + (\alpha_{02} + (a^2/r_L^2)\alpha_{02}^{(3)})\sin 2v_L + (a^2/r_L^2)(a_{04}^{(3)}\cos 4v_L + \alpha_{04}^{(3)}\sin 4v_L) + (a/r_L)(a_{01}\cos v_L + \alpha_{01}\sin v_L + a_{03}\cos 3v_L + \alpha_{03}\sin 3v_L)]$$

$$A_j^{(L)} = (a_L/r_L)^3[(a_{j0} + a_{j2}\cos 2v_L + \alpha_{j2}\sin 2v_L) + (a/r_L)(a_{j1}\cos v_L + \alpha_{j1}\sin v_L + a_{j3}\cos 3v_L + \alpha_{j3}\sin 3v_L)]$$

$$B_j^{(L)} = (a_L/r_L)^3[(b_{j0} + b_{j2}\cos 2v_L + \beta_{j2}\sin 2v_L) + (a/r_L)(b_{j1}\cos v_L + \beta_{j1}\sin v_L + b_{j3}\cos 3v_L + \beta_{j3}\sin 3v_L)]$$

$$j = 1, 2, 3$$

The coefficients a_{ji}, b_{ji}, α_{ji}, β_{ji}, depend on e, ω, i, and Ω in the following manner:

$$a_{00} = \tfrac{1}{8}(2 + e^2)(-1 + 3\cos^2 i) + \tfrac{9}{8}e^2\cos 2\omega \sin^2 i$$

$$a_{02} = [\tfrac{3}{8}(2 + e^2)\sin^2 i + \tfrac{9}{8}e^2\cos 2\omega(1 + \cos^2 i)]\cos 2\Omega - \tfrac{9}{4}e^2\sin 2\omega \cos i \sin 2\Omega$$

$$a_{10} = -\tfrac{1}{2}e(-1 + 3\cos^2 i) - \tfrac{3}{2}e\cos 2\omega \sin^2 i$$

$$a_{12} = -\tfrac{3}{2}e[\sin^2 i + \cos 2\omega(1 + \cos^2 i)]\cos 2\Omega + 3e\sin 2\omega \cos i \sin 2\Omega$$

$$a_{20} = \tfrac{1}{8}e^2(-1 + 3\cos^2 i) + \tfrac{3}{8}(2 - e^2)\cos 2\omega \sin^2 i$$

$$a_{22} = \tfrac{3}{8}[e^2\sin^2 i + (2 - e^2)\cos 2\omega(1 + \cos^2 i)]\cos 2\Omega - \tfrac{3}{4}(2 - e^2)\sin 2\omega \cos i \sin 2\Omega$$

$$a_{30} = a_{32} = 0$$

$$b_{10} = \tfrac{3}{2}e\sqrt{1 - e^2}\sin 2\omega \sin^2 i$$

$$b_{12} = \tfrac{3}{2}e\sqrt{1 - e^2}\sin 2\omega(1 + \cos^2 i)\cos 2\Omega + 3e\sqrt{1 - e^2}\cos 2\omega \cos i \sin 2\Omega$$

$$b_{20} = -\tfrac{3}{4}\sqrt{1 - e^2}\sin 2\omega \sin^2 i$$

$$b_{22} = -\tfrac{3}{4}\sqrt{1 - e^2}\sin 2\omega(1 + \cos^2 i)\cos 2\Omega - \tfrac{3}{2}\sqrt{1 - e^2}\cos 2\omega \cos i \sin 2\Omega$$

$$b_{30} = b_{32} = 0$$

$$a_{01} = \tfrac{3}{16}e(2 + \tfrac{1}{2}e^2)\xi - \tfrac{75}{32}e^3\eta \sin^2 i$$

$$a_{03} = -\tfrac{15}{16}e(2 + \tfrac{1}{2}e^2)\sigma \sin^2 i - \tfrac{25}{35}e^3\tau$$

$$a_{11} = -\tfrac{3}{16}(1 + \tfrac{11}{4}e^2)\xi + \tfrac{225}{64}e^2\eta \sin^2 i$$

$$a_{13} = \tfrac{15}{16}(1 + \tfrac{11}{4}e^2)\sigma\cdot\sin^2 i + \tfrac{75}{64}e^2\tau$$

$$a_{21} = \tfrac{3}{32}e(2 + e^2)\xi - \tfrac{45}{32}e(2 - e^2)\eta \sin^2 i$$

$$a_{23} = -\tfrac{15}{32}e(2 + e^2)\sigma \sin^2 i - \tfrac{15}{32}e(2 - e^2)\tau$$

$$a_{31} = -\tfrac{3}{64}e^2\xi + \tfrac{15}{64}(4 - 3e^2)\eta \sin^2 i$$

$$a_{33} = \tfrac{15}{64}e^2\sigma \sin^2 i + \tfrac{5}{64}(4 - 3e^2)\tau$$

$$b_{11} = -\tfrac{3}{64}\sqrt{1 - e^2}(4 + e^2)(\partial\xi/\partial\omega) + \tfrac{225}{64}e^2\sqrt{1 - e^2}\sin^2 i\,\tfrac{1}{3}(\partial\eta/\partial\omega)$$

$$b_{13} = \tfrac{15}{64}\sqrt{1 - e^2}(4 + e^2)\sin^2 i(\partial\sigma/\partial\omega) + \tfrac{75}{64}e^2\sqrt{1 - e^2}\,\tfrac{1}{3}(\partial\tau/\partial\omega)$$

$$b_{21} = \tfrac{3}{16}\sqrt{1 - e^2}(\partial\xi/\partial\omega) - \tfrac{45}{16}e\sqrt{1 - e^2}\sin^2 i\,\tfrac{1}{3}(\partial\eta/\partial\omega)$$

$$b_{23} = -\tfrac{15}{16}e\sqrt{1 - e^2}\sin^2 i(\partial\sigma/\partial\omega) - \tfrac{15}{16}e\sqrt{1 - e^2}\,\tfrac{1}{3}(\partial\tau/\partial\omega)$$

$$b_{31} = -\tfrac{3}{64}e^2\sqrt{1 - e^2}(\partial\xi/\partial\omega) + \tfrac{15}{64}(4 - e^2)\sqrt{1 - e^2}\sin^2 i\,\tfrac{1}{3}(\partial\eta/\partial\omega)$$

$$b_{33} = \tfrac{15}{64}e^2\sqrt{1 - e^2}(\partial\sigma/\partial\omega)\sin^2 i + \tfrac{5}{64}(4 - e^2)\sqrt{1 - e^2}\,\tfrac{1}{3}(\partial\tau/\partial\omega)$$

$$a_{00}^{(3)} = \tfrac{9}{512}(1 + 5e^2 + \tfrac{15}{8}e^4)(3 - 30\cos^2 i + 35\cos^4 i) + \tfrac{315}{64}e^2(1 + \tfrac{1}{2}e^2)(-\tfrac{1}{8} + \tfrac{7}{8}\cos^2 i)\cos 2\omega \sin^2 i + \tfrac{6615}{4096}e^4\cos 4\omega \sin^4 i$$

$$a_{02}^{(3)} = [\tfrac{15}{128}(1 + 5e^2 + \tfrac{15}{8}e^4)(-1 + 7\cos^2 i)\sin^2 i + \tfrac{105}{128}e^2(1 + \tfrac{1}{2}e^2)(1 - 6\cos^2 i + 7\cos^4 i)\cos 2\omega + \tfrac{2205}{1024}e^4\cos 4\omega(1 - \cos^4 i)]\cos 2\Omega + [\tfrac{105}{128}e^2(1 + \tfrac{1}{2}e^2)(5 - 7\cos^2 i)\sin 2\omega \cos i - \tfrac{2205}{512}e^4\sin 4\omega \sin^2 i\cdot\cos i]\sin 2\Omega$$

$$a_{04}^{(3)} = [\tfrac{105}{256}(1 + 5e^2 + \tfrac{15}{8}e^4)\sin^4 i + \tfrac{735}{512}e^2(1 + \tfrac{1}{2}e^2)\times\cos 2\omega(1 - \cos^4 i) + \tfrac{2205}{4096}e^4\cos 4\omega(1 + 6\cos^2 i + \cos^4 i)]\cos 4\Omega - \tfrac{735}{256}e^2(1 + \tfrac{1}{2}e^2)\sin 2\omega \sin^2 i \cos i + \tfrac{2205}{1024}e^4\sin 4\omega \cos i(1 + \cos^2 i)]\sin 4\Omega$$

$$a_{jl} = -\frac{1}{l}\frac{\partial}{\partial\Omega}a_{jl}$$

$$\beta_{jl} = -\frac{1}{l}\frac{\partial}{\partial\Omega}b_{jl}$$

$$\xi = \cos\omega(1 - 5\cos^2 i)\cos\Omega + \sin\omega(-11 + 15\cos^2 i)\sin\omega(-11 + 15\cos^2 i)\cos i \sin\Omega$$

$$\eta = \cos 3\omega \cos\Omega - \sin 3\omega \cos i \sin\Omega$$

$$\sigma = \cos\omega \cos 3\Omega - \sin\omega \cos i \sin 3\Omega$$

$$\tau = \cos 3\omega(1 + 3\cos^2 i)\cos 3\Omega - \sin 3\omega(3 + \cos^2 i)\cos i \times \sin 3\Omega$$

Substituting the expression for W_i in Eqs. [1], we see that each component in the right-hand parts of the equations may be represented as a product of two factors $F_1(M_i)$ and $F(M)$, the first of which is dependent only on the coordinate of the disturbing body and the second only on the coordinate of the satellite, i.e., $d\mathfrak{E}_2/dt = \Sigma F_1(M_i)\cdot F(M)$.

These equations cannot be integrated in the closed form. In this paper an integration by parts was used with the rejection of the small residual term:

$$\mathfrak{E}_2^{(i)} = \sum\int F_1(M_i)\cdot F(M)dt = \frac{1}{n}\sum I_1(M)\cdot F_1(M_i) - \frac{n_i}{n}\sum\int I_1(M)F_1'(M_i)dt = \frac{1}{n}\sum I_1(M)\cdot F_1(M_i) - \frac{n_i}{n^2}\sum F_1'(M_i)I_2(M) + \left(\frac{n_i}{n}\right)^2\sum\int I_2(M)\cdot F_1''(M_i)dt = \ldots$$

where

$$I_1(M) = \int F(M)dM \qquad I_2(M) = \int I_1(M)dM$$

$$F_1'(M_i) = \frac{d}{dM_i}[F_1(M_i)]$$

I_1 and I_2 can be integrated in closed forms. From the sequence of the forementioned equations, it is evident that at each stage of the integration by parts an integral appears in the form $\int I_l(M)F_1^{(l)}(M_i)dt$, i.e., the same as the original one but with a small multiplier $(n_i/n)^l$.

For the moon

$$n_L/n = 2.1\cdot 10^{-3}(a/c)^{3/2} \leqslant 6.7\cdot 10^{-2}$$

$$(n_L/n)^2 \leqslant 4.4\cdot 10^{-3}$$

For the sun

$$n_S/n = 1.6\cdot 10^{-4}(a/c)^{3/2} \leqslant 5.1\cdot 10^{-3}$$

in the region expressed in Eq. [3].

If one discards the integrals with the multipiers $(n_L/n)^2$ and (n_S/n), then in the expressions for the elements we disregard the components causing disturbances in the elements of the order of 10^{-9} for close satellites and of the order 10^{-7} for satellites having semimajor axes of the order of $a \approx 10c$.

In other words, in order correctly to obtain the order of the disturbances from the sun it is sufficient to integrate once by parts and to discard the remaining integral. In order to obtain the disturbances from the moon with acceptable accuracy, it is necessary to integrate twice by parts. However, it is also necessary to retain the most significant components from the integrals being discarded. These components follow from the constant and secular terms of the integrals I_1 and I_2, because the integration of these components yields large factors $1/n_i$ and $(1/n_i)^2$.

Therefore, let $F(M) = C + \Phi$ where C is a constant and Φ consists of periodic terms. Then $I_1 = CM + \Phi + C_1$ and

$$\frac{n_i}{n}\int I_1(M)F_1'(M_i)dt = \frac{1}{n}C_1F_1(M_i) + \frac{C}{n_i}\left[M_iF_1(M_i) - \int F_1(M_i)dM_i\right] + \frac{n_i}{n}\int \Phi\cdot F_1'(M_i)dt$$

Hence we see that the integrals obtained from the constant terms in I_1 are of the same order as the components $(1/n)I_1(M) \times F_1(M_i)$. However, the integrals due to secular terms in I_1 are much larger than terms being retained. This is due to the factor $1/n_i$ associated with these integrals. In the perturbation of the elements, these integrals give components of the first order relative to the small parameter δ_i since

$$\delta_i^2\frac{1}{n_i} = \frac{1}{n}\sqrt{\frac{m_i}{m_i + m_0}}\,\delta_i\left(\frac{c}{a}\right)^{3/2}$$

Consequently, the solution of Eqs. [1] will have the form

$$\mathfrak{E} = \mathfrak{E}_0 + \delta_L\mathfrak{E}_1^{(L)} + \delta_L^2\mathfrak{E}_2^{(L)} + \delta_S\mathfrak{E}_1^{(S)} + \delta_S^2\mathfrak{E}_2^{(S)}$$

Carrying out the integration by the indicated process we obtain

$$\frac{1}{n}\int\frac{\partial W_i}{\partial\mathfrak{E}}dt = \sqrt{\frac{m_i}{m_i + m_0}}\left(\frac{a}{c}\right)^{3/2}\delta_i\frac{1}{2}a^2\frac{\partial}{\partial\mathfrak{E}}\times \int\left(A_0^{(i)} - \frac{1}{2}eA_1^{(i)}\right)dM_i + \delta_i^2\left(\frac{a}{c}\right)^3\frac{1}{2}a^2\sum_{j=0}^{5}\frac{\partial}{\partial\mathfrak{E}}(\bar{A}_j^{(i)}\cos jE + \bar{B}_j^{(i)}\sin jE)$$

(for $\mathfrak{E} = \omega, i, \Omega$):

$$\frac{1}{n}\int\frac{\partial W_i}{\partial a}dt = \sqrt{\frac{m_i}{m_i + m_0}}\left(\frac{a}{c}\right)^{3/2}\delta_i\frac{\partial}{\partial a}\times\left[\frac{1}{2}a^2\int\left(A_0^{(i)} - \frac{1}{2}eA_1^{(i)}\right)dM_i\right] + \delta_i^2\left(\frac{a}{c}\right)^3\frac{\partial}{\partial a}\left[\frac{1}{2}a^2\sum_{j=0}^{5}(\bar{A}_j^{(i)}{}_i\cos jE + \bar{B}_j^{(i)}\sin jE)\right]$$

$$\frac{1}{n}\int\frac{\partial W_i}{\partial e}dt = \sqrt{\frac{m_i}{m_i + m_0}}\left(\frac{a}{c}\right)^{3/2}\delta_i\frac{1}{2}a^2\frac{\partial}{\partial e}\times\int\left(A_0^{(i)} - \frac{1}{2}eA_1^{(i)}\right)dM_i + \delta_i^2\left(\frac{a}{c}\right)^3\times\frac{1}{2}a^2\sum_{j=0}^{5}\left(\bar{\bar{A}}_j^{(i)} + \frac{\partial}{\partial e}\bar{A}_j^{(i)}\right)\cos_jE + \left(\bar{\bar{B}}_j^{(i)} + \frac{\partial}{\partial e}\bar{B}_j^{(i)}\right)\sin jE\Big]$$

$$\frac{1}{n}\int\frac{\partial W_i}{\partial M}dt = \delta_i^2\left(\frac{a}{c}\right)^3\frac{1}{2}a^2\times\sum_{j=0}^{4}(\tilde{A}_j^{(i)}\cos jE + \tilde{B}_j^{(i)}\sin jE)$$

The coefficients $\tilde{A}_j^{(i)}$, $\tilde{B}_j^{(i)}$, $\bar{A}_j^{(i)}$, $\bar{B}_j^{(i)}$, $\bar{\bar{A}}_j^{(i)}$, $\bar{\bar{B}}_j^{(i)}$ are obtained from the coefficients $A_j^{(i)}$ and $B_j^{(i)}$ of the disturbing functions according to the following formula (for simplification of the entry the lower index i in all coefficients is omitted).

$$\bar{A}_0 = -\tfrac{1}{2}e(B_1 - \tfrac{1}{2}eB_2) + \tfrac{1}{2}(n_i/n)e[(1 - \tfrac{3}{8}e^2)A_1' - \tfrac{3}{4}eA_2' + \tfrac{1}{8}e^2A_3']$$

$$\bar{A}_1 = -(B_1 - \tfrac{1}{2}eB_3) + (n_i/n)[(1 - \tfrac{3}{8}e^2)A_1' - \tfrac{3}{4}eA_2' + \tfrac{1}{8}e^2A_3']$$

$$\bar{A}_2 = \tfrac{1}{2}(\tfrac{1}{2}eB_1 - B_2 + \tfrac{1}{2}eB_3) + (n_i/n)[(\tfrac{1}{8}e^3 - \tfrac{3}{8}e)A_1' + (\tfrac{1}{4} + \tfrac{1}{6}e)A_2' - \tfrac{5}{24}eA_3']$$

$$\bar{A}_3 = \tfrac{1}{3}(\tfrac{1}{2}eB_2 - B_3) + (n_i/n)[\tfrac{1}{24}e^2A_1' - \tfrac{5}{36}eA_2' + (\tfrac{1}{9} + \tfrac{1}{16}e^2)A_3']$$

$$\bar{A}_4 = \tfrac{1}{8}eB_3 + (n_i/n)(\tfrac{1}{48}e^2A_2' - \tfrac{7}{96}eA_3')$$

$$\bar{A}_5 = \tfrac{1}{80}(n_i/n)e^2A_3'$$

$$\bar{B}_1 = A_1(1 - \tfrac{1}{2}e^2) - \tfrac{1}{2}eA_2 - (n_i/n)[(-1 + \tfrac{3}{8}e^2)B_1' + (\tfrac{3}{4}e - \tfrac{1}{4}e^3)B_2' - \tfrac{1}{8}e^2B_3']$$

$$\bar{B}_2 = \tfrac{1}{2}(A_2 - \tfrac{1}{2}eA_1 - \tfrac{1}{2}eA_3) - (n_i/n)[\tfrac{3}{8}eB_1' - (\tfrac{1}{4} + \tfrac{1}{6}e^2)B_2' + \tfrac{5}{24}eB_3']$$

$$\bar{B}_3 = \tfrac{1}{3}(A_3 - \tfrac{1}{2}eA_2) - (n_i/n)[-\tfrac{1}{24}e^2B_1' + \tfrac{5}{36}eB_2' - (\tfrac{1}{9} + \tfrac{1}{16}e^2)B_3']$$

$$\bar{B}_4 = -\tfrac{1}{8}eA_3 - (n_i/n)[\tfrac{7}{96}eB_3' - \tfrac{1}{48}e^2B_2']$$

$$\bar{B}_5 = \tfrac{1}{80}(n_i/n)e^2B_3'$$

$$\bar{\bar{A}}_0 = \tfrac{1}{2}B_1 + (n_i/n)[(-\tfrac{1}{2} + \tfrac{1}{4}e^2)A_1' + \tfrac{1}{4}eA_2']$$

$$\bar{\bar{A}}_1 = \tfrac{1}{2}B_2 + (n_i/n)(\tfrac{1}{8}eA_1' - \tfrac{1}{4}A_2' + \tfrac{1}{8}eA_3')$$

$$\bar{\bar{A}}_2 = -\tfrac{1}{2}B_1 + \tfrac{1}{2}B_3 + (n_i/n)[(-\tfrac{1}{4}e^2 + \tfrac{1}{2})A_1' - \tfrac{1}{6}eA_2' - \tfrac{1}{6}A_3']$$

$$\bar{\bar{A}}_3 = -\tfrac{1}{2}B_2 + (n_i/n)[-\tfrac{1}{8}eA_1' + \tfrac{1}{4}A_2' - \tfrac{1}{16}eA_3']$$

$$\bar{\bar{A}}_4 = -\tfrac{1}{2}B_3 + (n_i/n)[-\tfrac{1}{12}eA_2' + \tfrac{1}{6}A_3']$$

$$\bar{\bar{A}}_5 = -\tfrac{1}{16}(n_i/n)eA_3'$$

$$\bar{\bar{B}}_1 = \tfrac{1}{2}eA_1 - \tfrac{1}{2}A_2 + (n_i/n)[\tfrac{5}{8}eB_1' - \tfrac{1}{4}(1 + e^2)B_2' + \tfrac{1}{8}eB_3']$$

$$\bar{\bar{B}}_2 = \tfrac{1}{2}A_1 - \tfrac{1}{2}A_3 + (n_i/n)[\tfrac{1}{2}B_1' - \tfrac{1}{6}eB_2' - \tfrac{1}{6}B_3']$$

$$\bar{\bar{B}}_3 = \tfrac{1}{2}A_2 + (n_i/n)[-\tfrac{1}{8}eB_1' + \tfrac{1}{4}B_2' - \tfrac{1}{16}eB_3']$$

$$\bar{\bar{B}}_4 = \tfrac{1}{2}A_3 + (n_i/n)[-\tfrac{1}{12}eB_2' + \tfrac{1}{6}B_3']$$

$$\bar{\bar{B}}_5 = -\tfrac{1}{16}(n_i/n)eB_3'$$
$$\tilde{A}_0 = \tfrac{1}{2}eA_1 + (n_i/n)(\tfrac{1}{2}eB_1' - \tfrac{1}{4}e^2B_2')$$
$$\tilde{A}_1 = A_1 + (n_i/n)(B_1' - \tfrac{1}{2}eB_2')$$
$$\tilde{A}_2 = A_2 + (n_i/n)(-\tfrac{1}{4}eB_1' - \tfrac{1}{2}B_2' - \tfrac{1}{4}eB_3')$$
$$\tilde{A}_3 = A_3 + (n_i/n)(\tfrac{1}{3}B_3' - \tfrac{1}{6}eB_2')$$
$$\tilde{A}_4 = -\tfrac{1}{8}(n_i/n)eB_3'$$
$$\tilde{B}_1 = B_1 + (n_i/n)[-(1 - \tfrac{1}{2}e^2)A_1' + \tfrac{1}{2}eA_2']$$
$$\tilde{B}_2 = B_2 + (n_i/n)[\tfrac{1}{4}eA_1' - \tfrac{1}{2}A_2' + \tfrac{1}{4}eA_3']$$
$$\tilde{B}_3 = B_3 + (n_i/n)[\tfrac{1}{6}eA_2' - \tfrac{1}{3}A_3']$$
$$\tilde{B}_4 = \tfrac{1}{8}(n_i/n)eA_3'$$
$$A_j' = \partial A_j/\partial M_i$$
$$B_j' = \partial B_j/\partial M_i$$

The integrals $\int A_j^{(i)}dM_i$ are integrable in closed form if the variable of the integration is the true anomaly of the disturbing body v_i. Then

$$\int A_{(i)}{}^j dM_i = C_{(i)}{}^j v_i + \sum_{l=1}^{7} (\Xi_{(i)}{}^{jl} \cos lv_i + H_{(i)}{}^{jl} \sin lv_i)$$

The coefficients $C_j^{(i)}$, $\Xi_{jl}^{(i)}$, $H_{jl}^{(i)}$ are dependent on the coefficients $A_j^{(i)}$, $B_j^{(i)}$ and on the parameters of the orbit of the disturbing body (the semimajor axis and eccentricity). The calculation of these coefficients is not difficult; because of lack of space they are not carried out here.

With the help of these integrals it is easy to obtain the expression for $\mathfrak{E}_2^{(S)}$ and $\mathfrak{E}_2^{(L)}$ from Eqs. [1]. All coefficients are highly simplified because, during the investigation of the disturbances due to the sun, we take into account in the disturbing function only one component with P_2 and integrate by parts only once.

However, the formulas for the elements are quite complex. Here the principal parts of the perturbations in the elements are given, which are independent of the eccentricity and parallax of the disturbing body:

$$a_{\text{pr}}{}^{(i)} = 0$$

$$e_{\text{pr}}{}^{(i)} = \sqrt{\frac{m_i}{m_i + m_0}}\,\delta_i \left(\frac{a}{c}\right)^{3/2} \frac{15}{16}\, e\sqrt{1 - e^2} \times [-\sin 2\omega(1 + \cos^2 i)(\sin 2\tilde{\Omega} - \sin 2\tilde{\Omega}^{(0)}) + 2\cos 2\omega \cos i\,(\cos 2\tilde{\Omega} - \cos 2\tilde{\Omega}^{(0)})] + \frac{15}{8}\,\delta_i^2 e\sqrt{1 - e^2} \times (a/c)^3 \sin 2\omega \sin^2 i(M - M^{(0)})$$

$$i_{\text{pr}}{}^{(i)} = \sqrt{\frac{m_i}{m_i + m_0}}\,\delta_i \left(\frac{a}{c}\right)^{3/2} \frac{3}{16}\frac{\sin i}{\sqrt{1 - e^2}} \times [-5e^2 \sin 2\omega \cos i\,(\sin 2\tilde{\Omega} - \sin 2\tilde{\Omega}) + (2 + 3e^2 + 5e^2 \cos 2\omega)(\cos 2\tilde{\Omega} - \cos 2\tilde{\Omega}^{(0)})] - \frac{15}{16}\,\delta_i^2 \frac{e^2}{\sqrt{1 - e^2}} \left(\frac{a}{c}\right)^3 \sin 2\omega \sin 2i(M - M^{(0)})$$

$$\omega_{\text{pr}}{}^{(i)} = \sqrt{\frac{m_i}{m_i + m_0}}\,\delta_i \left(\frac{a}{c}\right)^{3/2} \frac{3}{16}\frac{1}{\sqrt{1 - e^2}} \times \{[-(1 - e^2)(3 + 5\cos 2\omega) + 5(1 - \cos 2\omega)\cos^2 i](\sin 2\tilde{\Omega} - \sin 2\tilde{\Omega}^{(0)}) + 5(-2 + e^2)\sin 2\omega \cos i(\cos 2\tilde{\Omega} - \cos 2\tilde{\Omega}^{(0)})\} + \delta_i^2 \left(\frac{a}{c}\right)^3 \frac{3}{8}\frac{1}{\sqrt{1 - \rho^2}} [(1 - e^2)(-1 + 5\cos 2\omega) + 5(1 - \cos 2\omega)\cos^2 i](M - M^{(0)})$$

$$\Omega_{\text{pr}}{}^{(i)} = \sqrt{\frac{m_i}{m_i + m_0}}\,\delta_i \left(\frac{a}{c}\right)^{3/2} \frac{3}{16}\frac{1}{\sqrt{1 - e^2}} \times [-(2 + 3e^2 - 5e\cos 2\omega) \times \cos i\,(\sin 2\tilde{\Omega} - \sin 2\tilde{\Omega}^{(0)}) + 5e^2 \sin 2\omega(\cos 2\tilde{\Omega} - \cos 2\tilde{\Omega})] + \delta_i^2 \left(\frac{a}{c}\right)^3 \frac{3}{8}\frac{(-2 - 3e^2 + 5e^2\cos 2\omega)}{\sqrt{1 - e^2}} \cos i(M - M^{(0)})$$

$$M_{\text{pr}}{}^{(i)} = \sqrt{\frac{m_i}{m_i + m_0}}\,\delta_i \left(\frac{a}{c}\right)^{3/2} \times \left\{\frac{15}{8}(1 + e^2)\sin 2\omega \cos i(\cos 2\tilde{\Omega} - \cos 2\tilde{\Omega}^{(0)}) + \frac{3}{16}[(7 + 3e^2)\sin^2 i + 5(1 + e^2)\cos 2\omega(1 + \cos^2 i)](\sin 2\tilde{\Omega} - \sin 2\tilde{\Omega}^{(0)})\right\} - \delta_i^2 \left(\frac{a}{c}\right)^3 \left\{\frac{3}{2}C_a + \frac{1}{8}(7 + 3e^2) \times (-1 + 3\cos^2 i) + \frac{15}{8}(1 + e^2)\cos 2\omega \sin^2 i\right\}(M - M^{(0)})$$

$$\tilde{\Omega} = \Omega - M_i \qquad \tilde{\Omega}^{(0)} = \Omega - M_i^{(0)}$$

C_a is the constant of integration in the element a:

$$a = a_0\left[1 + \delta_i^2(a/c)^3 \sum_{j=0}^{4} (C_a + \tilde{A}_j \cos jE + \tilde{B}_j \sin jE)\right]$$

Using C_a it is possible to make the secular part in M equal to zero. $M_i^{(0)}$ and $M^{(0)}$ are the initial values of the mean anomalies of the disturbing body and the satellite.

In the expressions for $\mathfrak{E}_{\text{pr}}{}^{(i)}$ the terms with δ_i give long periodic variations of the elements of the satellite orbit during a period of revolution of the disturbing body.

Let us examine the maximum variation of the elements caused by these disturbances, i.e., the difference between their maximum and minimum amplitudes. We shall designate this difference by $\bar{A}\,\mathfrak{E}^{(i)}$:

$$\bar{A}_e{}^{(L)} = 5\cdot 10^{-5}(a/c)^{3/2}$$

for

$$e = (1/\sqrt{2}) \qquad i = 0$$

For $a = 9c$, i.e., for $a \approx 58{,}000$ km, we have $A_e{}^{(L)} = 1.4\cdot 10^{-3}$; the oscillations of the height of the perigee amount to approximately 80 km. For the other elements we have

$$\bar{A}_\pi{}^{(L)} = 10'.8$$

for $\quad i = 0 \quad \omega = \pi/2 \quad e = 0 \quad a = 9c$

$$\bar{A}_i{}^{(L)} = 2'$$

for $\quad i = \pi/2 \quad a = 9c \quad e = 0$

$$\bar{A}_\Omega{}^{(L)} = 2'$$

for $\quad i = 0 \quad a = 9c \quad e = 0$

The maximum variations of the elements caused by the secular lunar perturbations over one lunar period are

$$[e^{(L)}]_{\max} = 24.5\cdot 10^{-3}$$

for $\quad \omega = \pi/4 \quad i = \pi/2 \quad e = 1/\sqrt{2} \quad a = 9c$

This reduces the perigee by 240 km over the period of revolution of the moon:

$$[i^{(L)}]_{\max} = 9'$$

for $\quad i = \pi/2 \quad \omega = \pi/4 \quad e = 1/\sqrt{2} \quad a = 9c$

$$[\omega^{(L)}]_{\max} = 32'$$

for $\quad e^2 = \tfrac{1}{2} \quad i = 0 \quad a = 9c$

$$[\Omega^{(L)}]_{\max} = 16'$$

for $e^2 = \frac{1}{2}$ $a = 9c$ $i = 0$

The perturbations of the elements $i^{(L)}$, $\omega^{(L)}$, $\Omega^{(L)}$ increase strongly when $e \to 1$. More detailed formulas will be published in the future.

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References

1 Subbotin, M. F., *A Course in Celestial Mechanics* (Moscow-Leningrad, 1937), Vol. 2, p. 39.

2 Subbotin, M. F., *A Course in Celestial Mechanics* (Moscow-Leningrad, 1941), Vol. 1, p. 7.

3 Subbotin, M. F., *A Course in Celestial Mechanics* (Moscow-Leningrad, 1937), Vol. 2, p. 323.

Reviewer's Comment

The article treats the motion of an artificial satellite around the spherical earth under the influence of the moon and the sun. Since the coupling effects among the oblateness of the earth and the attraction of the moon and of the sun are neglected, the problem is similar to the lunar theory.

However, in contrast with the lunar theory, expansion of the disturbing function in powers of the eccentricity of the satellite is inefficient or impossible due to a possible large value of the eccentricity. The article avoids this difficulty by carrying out integration with a method of "integration by parts," the eccentric anomaly of the satellite being the variable of integration, and the true anomaly of the moon or the sun, the variable of differentiation. This may be the point of the present article.

The lunar or solar perturbations on the satellite motion are evaluated with the orbital plane of the moon or the ecliptic as the reference plane, respectively, so that Ω, i, and ω should be distinguished in the two cases, and, moreover, when both the lunar and solar perturbations are combined, transformations to a common reference plane are necessary.

Since the main source of perturbations is the oblateness of the earth, the reference plane should be the equatorial plane of the earth even if perturbations due to the oblateness and the attractions of the moon and of the sun are evaluated separately. This may constitute a weakness of the theory.

The article takes into account the influence of the moon through the fourth harmonic of the disturbing function. To preserve the order of the accuracy of the theory, the main perturbation of the moon's coordinates due to the solar attraction should be considered.

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Dosimetric Measurements on the Second Soviet Spaceship Satellite

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1 Introduction

In order to insure the safety of astronauts, a detailed knowledge of the physical properties of cosmic space is indispensable. Study of cosmic radiation, including the following forms of radiation, is of particular importance:

1) Charged particle flux (protons with energies $E_p > 10^8$ ev, in particular), penetrating into the solar system from the galaxy, which is isotropic in space and nearly uniform in time. This form of radiation which, in its narrow sense, constitutes cosmic rays, has been known for a long time and has been comparatively well studied. The cosmic ray intensity in interplanetary space during the years of maximum solar activity constitutes 2 to 2.5 particles$\cdot$cm$^{-2}\cdot$sec^{-1} (1).[1] As the solar activity decreases, the intensity of cosmic rays is doubled (2).

2) Charged particles (protons with energies of about 10^8 ev) and γ quanta, whose appearance is linked with chromospheric flares on the sun. Most of these flares are observed during the maximum period of the 11-year cycle of solar activity. During the last years, several flares were observed after which an increase in the proton flux of 10^3 and higher took place in the near-earth space (3).

3) Radiation originating from the earth's radiation belts. At present the existence of two such fundamental radiation belts has been established (4).[2] The outer belt is at a distance of 13,000 to 50,000 km from the earth in the equatorial plane. It consists basically of electrons with mean energy of the order of 10^5 ev. According to measurements on cosmic rockets, the maximum radiation intensity in the outer belt corresponds to the power of a dose to 10 r/hr (under an average substance layer exceeding 1 g$\cdot$cm^{-2} thickness) (5). The processes linked with chromospheric flares in the sun have a substantial influence on the intensity and position of the outer radiation belt (5). The inner radiation belt is situated at distances from 600 to 4500 km from the surface of the earth (in the magnetic equator plane), and it contains, besides electrons, protons with energies of the order of 10^8 ev. The radiation intensity in this belt is also high; it reaches 10 r/hr for the dose power under 1 g$\cdot$cm^{-2} substance layer (5, 6).[3]

The existence of the forementioned types of radiation, which may present a serious danger for biological objects under specific conditions, requires the carrying out of dosimetric control aboard spacecraft. This requirement becomes particularly obvious if one considers possible rapid variations in intensity of either form of cosmic radiation caused by solar processes.

The second Soviet spaceship satellite was launched on August 19, 1960. Its orbit plane was inclined at an angle of

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[1] Numbers in parentheses indicate References at end of paper.

[2,3] See Editor's Note.